Solutions

7.4: Probability and Counting Techniques

Definition 1. A probability distribution in which all outcomes are equally likely is called the uniform distribution. In this case, if the sample space has n elements; i.e. $S = \{s_1, s_2, \ldots, s_n\}$, then

$$P(s_i) = \frac{1}{n} \quad \text{and} \quad P(E) = \frac{n(E)}{n(S)}$$

for any event $E \subseteq S$.

Example 1.

(a) Flipping a fair coin yields the uniform distribution with 2 outcomes.

(b) Rolling a fair die yields the uniform distribution with 6 outcomes.

(c) Drawing a five-card poker hand yields the uniform distribution with $C(52, 5) = 2,598,960$ outcomes.

(d) Anytime an experiment consists of "selecting at random" we assume the uniform distribution is being used.

Example 2. A bag contains four red marbles and two green ones. Upon seeing the bag, Suzan sticks her hand in and grabs three at random. Find the probability that she will get both green marbles.

Sample Space has $C(6,3) = 20$ outcomes with uniform distribution.

# of outcomes containing both green marbles = Step 1: Choose green $C(2,2) = 1$; Step 2: Choose 1 red $C(4,1) = 4$ $\Rightarrow$ 4 outcomes. So

Example 3. You are dealt 5 cards from a well-shuffled deck of 52. Find the probability that you have a full house. $P(2\text{ greens}) = \frac{4}{20} = 0.2$

Sample space has $C(52,5) = 2,598,960$ outcomes uniformly distributed.

Let FH be the event you have a full house.

Step 1: Choose denomination of set of 3 $C(13,1) = 13$

Step 2: Choose 3 of the 4 suits $C(4,3) = 4$

Step 3: Choose denomination of the pair $C(12,1) = 12$

Step 4: Choose 2 of the 4 suits $C(4,2) = 6$

Thus $n(FH) = 13 \cdot 4 \cdot 12 \cdot 6 = 3,744$. Thus $P(FH) = \frac{n(FH)}{n(S)} = \frac{3,744}{2,598,960} = 0.001441$

Example 4. You are playing poker, and you have been dealt the following hand:

$$J\spadesuit, J\diamondsuit, J\heartsuit, 2\clubsuit, 10\spadesuit.$$

You decide to exchange the last two cards. The exchange works as follows: The two cards are discarded (not replaced in the deck), and you are dealt two new cards.

(a) Find the probability that you end up with a full house.

(b) Find the probability that you end up with four jacks.

(c) What is the probability that you end up with either a full house or four jacks?

S = sets of 2 cards chosen from remaining 47

So $n(S) = C(47,2) = 1081$.

(a) Alt 1: 2's or 10's

Step 1: $C(2,1)$ ·

Step 2: $C(3,2)$

$2 \cdot 3$ +

Alt 2: Not 2's, 10's or J's

Step 1: $C(10,1)$

Step 2: $C(4,2)$

$10 \cdot 6 = 66$.

So $P(FH) = \frac{66}{1081} = 0.0611$

(b) Step 1: Choose Jack $C(1,1) = 1$

Step 2: Choose anything $C(46,1) = 46$

So $P(4\text{ Jacks}) = \frac{46}{1081} = 0.0426$.

(c) $P(FH \cup 4\text{ Jacks}) = \frac{66+46}{1081} = 0.1036$.

$0.0611 + 0.0426 = 0.1037$ ✓

What is going on here?

Example 5. The Student Affairs Committee at The Tolkien School of Wizardcraft and Hobbitry shall consist of one elected faculty senator, one faculty senator-at-large, one elected student senator, five student senators-at-large (including one from the graduate school), two delegates from the Student Government Association, the President of the Student Government Association or his/her designee, and the President of the Graduate Student Organization.

You are an undergraduate student and, even though you are not an elected student senator, you would very much like to serve on the Student Affairs Committee. The senators-at-large as well as the Student Government delegates are chosen by means of a random drawing from a list of candidates. There are already 13 undergraduate candidates for the position of senator-at-large, and 6 candidates for Student Government delegates, and you have been offered a position on the Student Government Association by the President (who happens to be a good friend of yours), should you wish to join it. (This would make you ineligible for a senator-at-large position.) What should you do?

There are four undergraduate senator-at-large positions and two Student Government Associate positions.

This position gives $C(14,4) = 1001$ options and $C(1,1) \cdot C(13,3) = 286$ options include you.

$P(E) = \frac{286}{1001} = \frac{2}{7} \approx 0.2857$

This position gives $C(7,2) = 21$ options and $C(1,1) \cdot C(6,1) = 6$ options include you.

$P(E) = \frac{6}{21} = \frac{2}{7} \approx 0.2857$